Decision Theory, with Applications to Epistemology: **Formal Skills Assignment**

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1 Binary relations and preferences

1.1 What is wrong with the following argument to the conclusion that reflexivity of a binary relation R is a consequence of transitivity and symmetry of R:

> If $\langle x, y\rangle \in R$, then, by symmetry, $\langle y, x\rangle \in R$, and hence by transitivity, $\langle x, x\rangle \in R$?

Terminology: R is reflexive iff $\langle x, x\rangle \in R$; R is transitive iff $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ implies $\langle x, z\rangle \in R$; R is symmetric iff $\langle x, y\rangle \in R$ implies $\langle y, x\rangle \in R$.

1.2 Prove that

> If R is negatively transitive and asymmetric, then it is irreflexive, transitive and acyclic.

Terminology: R is negatively transitive iff $\langle x, y\rangle \notin R$ and $\langle y, z\rangle \notin R$ implies $\langle x, z\rangle \notin R$; R is asymmetric iff $\langle x, y\rangle \in R$ implies $\langle y, x\rangle \notin R$; R is irreflexive iff $\langle x, x\rangle \notin R$; R is acyclic iff $\langle x_1, x_2\rangle \in R$, $\langle x_2, x_3\rangle \in R$, $\dots$, $\langle x_{n-1}, x_n\rangle \in R$ implies $x_1 \neq x_n$.

1.3 Houthakker's axiom (aka the Weak Axiom of Revealed Preference) is a well known constraint on choice functions, equivalent to the conjunction of Sen's properties α and β:

> If $x, y \in \mathcal{A}$, $x, y \in \mathcal{B}$, $x \in c(\mathcal{A})$, and $y \in c(\mathcal{B})$, then $x \in c(\mathcal{B})$. (where $c(\mathcal{A}) = \{x \mid x \in \mathcal{A} \wedge \forall y(y \in \mathcal{A} \to \langle y, x\rangle \notin R)\}$, with R being a preference relation on $\mathcal{A} \cup \mathcal{B}$)

Prove that Houthakker's axiom is satisfied if R is asymmetric and negatively transitive.

2 Probability and random variables

2.1 Prove the following theorems, using the standard Kolmogorov axioms

1. $\Pr(H \mid E) > \Pr(H)$ (i) iff $\Pr(H \mid E) > \Pr(H \mid \overline{E})$, (ii) iff $\Pr(H \cap E) > \Pr(H)P(E)$ and (iii) iff $\Pr(E \mid H) > \Pr(E \mid \overline{H})$.

2. If $\Pr(H_1 \cap H_2) = 0$, then $\Pr(E \mid H_1) > \Pr(E \mid H_2)$ iff $\Pr(H_1 \mid E \wedge (H_1 \vee H_2)) > \Pr(H_1 \mid H_1 \vee H_2)$.

3. If (i) $\Pr(B \mid A \cap C) > \Pr(B \mid C)$, (ii) $\Pr(B \mid A) \leq \Pr(B)$, and (iii) $B \subseteq C$, then (iv) $\Pr(C \mid A) \leq \Pr(C)$.

2.2 Suppose a box contains five coins, and that for each coin there is a different probability that a head will be obtained when a coin is tossed. let $\Pr_i$ denote the probability of a head when the ith coin is tossed ($1 \leq i \leq 5$), and suppose that $\Pr_1 = 0$, $\Pr_2 = 1/4$, $\Pr_3 = 1/2$, $\Pr_4 = 3/4$ and $\Pr_5 = 1$.

1. Suppose that one coin is selected at random from the box and when it is tossed once, a head is obtained. What is the posterior probability that the ith coin was selected ($1 \leq i \leq 5$)?

2. If the same coin were tossed again, what would be the probability of obtaining another head?

3. If a tail had been obtained on the first toss of the selected coin and the same coin were tossed again, what would be the probability of obtaining a head on the second toss?

2.3 (a) The cumulative distribution function (cdf) of a random variable X is a function F from the reals such that $F(x) = \Pr(X \leq x)$. This function is non-decreasing as x increases. Why so?
(b) Suppose that a discrete random variable X has the following probability mass function (pmf), where c is a constant:

$$f(x) = \begin{cases} cx & \text{if } 1 \leq \text{x} \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Determine the value of c.
(c) Suppose that two fair dice are rolled, and let X denote the absolute value of the difference between the two numbers that appear.

1. Determine and sketch X's (i) pmf, (ii) cdf.

2. Calculate the expected value of X.

3 Utilities

3.1 Suppose an agent has preferences for paired items $\langle x, y\rangle$, where item x is the amount of time she has to live (in years) and y is the amount of time she will go through her death throes (in days). Let us suppose that x varies continuously between 0 and 20 and y between 0 and 14. The agent always prefers to live longer and suffer less but puts an absolute priority on living longer (i.e. no reduction in y will compensate for a reduction in x). Explain why it is impossible to use an ordinary finite ordinal utility function to represent the agent's preferences, even if her preference ordering is both complete and transitive.

3.2 Using the 'existence' part of the proof of the VNM representation theorem given in Resnik (pp. 93–96), show that, for all numbers a and lotteries x, y

1. $L(1, x, y) \sim x$
2. $L(0, x, y) \sim y$
3. $L(a, x, y) \sim L((1 - a), y, x)$
4. $L(a, x, x) \sim x$

3.3 Let B (for 'best')denote an arbitrary basic prize such that no other basic prize is strictly preferred to it. Reasoning directly from the 'rationality conditions' mentioned in Resnik's version of the VNM representation theorem (namely: his 'ordering condition', 'continuity condition', 'better-prizes condition', 'better-chances condition' and 'reduction of compound lotteries condition'), prove that:

1. There is no number a, or basic prize $x \neq B$ for which $L(a, x, B) \succ L(a, B, B)$ or $L(a, B, x) \succ L(a, B, B)$.
2. There is no number a, or basic prizes $x, y \neq B$ for which $L(a, x, y) \succ L(a, B, B)$.

3.4 Now define the 'degree' of a lottery as follows:

1. All basic prizes are of degree 0.
2. If n is the maximum of the degrees of L_1 and L_2, then the degree of $L(a, L_1, L_2)$ is $n + 1$.

Suppose no lottery of degree $< n$ is preferred to B. Again, reasoning directly from the aforementioned 'rationality conditions', show that

1. There is no number a, and lottery L of degree $< n$ for which $L(a, B, L) \succ L(a, B, B)$ or $L(a, L, B) \succ L(a, B, B)$.
2. There is no number a, and lotteries L_1, L_2 of degree $< n$ for which $L(a, L_1, L_2) \succ L(a, B, B)$.

It follows from this and the previous exercise that no lottery of degree > 0 if preferred to $L(a, B, B)$, for any number a.

3.4 Reasoning directly from the 'rationality conditions',

1. Show that for no number a, $L(a, B, B) \succ B$. Hint: apply the conclusion of exercise 3.3 to $L(a, L(a, B, B), L(a, B, B))$.

2. Show that for no number a, $B \succ L(a, B, B)$.