REASONS TO BELIEVE AND REASONS TO NOT
Technical Details

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1. Preliminaries

$\mathcal{L}$: formal language that includes
- $\mathcal{L}_P$: arbitrary propositional ('factual') language, constructed by means of the standard Boolean connectives $\{\vee, \wedge, \neg, \rightarrow\}$
- $\mathcal{L}_E$: smallest extension of $\mathcal{L}_P$ that includes all sentences of the form $\varphi \rhd \psi$ and $\varphi \bowtie \psi$ (where $\varphi \in \mathcal{L}_P$ and $\psi \in \mathcal{L}_E$)

K: arbitrary belief set, a subset of $\mathcal{L}$

$\mathbb{K}$: set of all rationally permissible belief sets

Cn: consequence operator, mapping subsets of $\mathcal{L}$ onto their set of logical consequences.

$*/\dot{-}$: revision/contraction functions from $\mathbb{K} \times \mathcal{L}$ to $\mathbb{K}$

2. Principles[1]

Belief sets:

(Cl) $\mathrm{Cn}(K) \subseteq K$

(Con) There is no finite $\Gamma \subseteq K$ such that $A \wedge \neg A \in \mathrm{Cn}(\Gamma)$.

Revision:

(AGM*2) If $\neg A \notin \mathrm{Cn}(\varnothing)$, then $A \in K * A$.

(AGM*3) If $B \in K * A$, then $B \in \mathrm{Cn}(K \cup \{A\})$

(AGM*4) If $\neg A \notin K$ and $B \in \mathrm{Cn}(K \cup \{A\})$, then $B \in K * A$

(AGM*V) If $\neg A \in \mathrm{Cn}(\varnothing)$ then $K * A = K$.

(AGM*6) If $\mathrm{Cn}(A) = \mathrm{Cn}(B)$, then $K * A = K * B$

[1]Unless otherwise noted, all principles are to be read as holding for all $K \in \mathbb{K}$ and $A, B, C \in \mathcal{L}_P$.

(AGM*7) If $C \in K * (A \wedge B)$ then $C \in \mathrm{Cn}(K * A \cup \{B\})$

(AGM*8) If $\neg B \notin K * A$ and $C \in \mathrm{Cn}((K * A) \cup \{B\})$, then $C \in K * (A \wedge B)$

(CM) If $B \in K * A$ and $C \in K * A$, then $C \in K * A \wedge B$

(DP*1) If $C \in \mathrm{Cn}(A)$, then $B \in K * A$ iff $B \in (K * C) * A$

(DP*2) If $\neg C \in \mathrm{Cn}(A)$, then $B \in K * A$ iff $B \in (K * C) * A$

(DP*3) If $B \in K * A$, then $B \in (K * B) * A$

(DP*4) If $\neg B \notin K * A$, then $\neg B \notin (K * B) * A$

(SVAC*) If $A \in K$, then $(K * A) * B = K * B$[2]

Contraction:

(AGM $\dot{-}$2) If $B \in K \dot{-} A$, then $B \in K$

(AGM $\dot{-}$3) If $A \notin K$ or $A \in \mathrm{Cn}(\emptyset)$, then $K \dot{-} A = K$

(AGM $\dot{-}$4) If $A \notin \mathrm{Cn}(\emptyset)$, then $A \notin K \dot{-} A$

(AGM $\dot{-}$5) If $\mathrm{Cn}(A) = \mathrm{Cn}(B)$, then $K \dot{-} A = K \dot{-} B$

(AGM $\dot{-}$6) If $B \in K$, then $B \in \mathrm{Cn}((K \dot{-} A) \cup \{A\})$

(DP $\dot{-}$2) If $C \in \mathrm{Cn}(A)$, then $B \in K * A$ iff $B \in (K \dot{-} C) * A$

(DP $\dot{-}$4) If $B \notin K * A$, then $B \notin (K \dot{-} B) * A$

(DP $\dot{-}$4+) If $B \in \mathrm{Cn}(C)$, then if $B \notin (K \dot{-} C) * A$, then $B \notin (K \dot{-} B) * A$

(SVAC $\dot{-}$) If $A \notin K$, then $(K \dot{-} A) * B = K * B$

Revision/contraction:

(DPf3) If $A \in f(K)$, then $A \in f(K * A)$

Evidential beliefs:[3]

(Sup-) $A \rhd B \in K$ iff $B \in K * A$

(Und-) $A \bowtie B \in K$ iff $B \notin K * A$

(Sup) $A \rhd B \in K$ iff $B \in (K \dot{-} B) * A$

(Und) $A \bowtie B \in K$ iff $B \notin (K * B) * A$

[2]Note: this is assumed to hold for all $A \in \mathcal{L}$

[3]All principles are to be read as holding for all $K \in \mathbb{K}$, $A \in \mathcal{L}_P$ and $A \in \mathcal{L}_E$.

3. Observations and selected proofs

Observation 1. *Given (Cl), (Con) (AGM*3) and (AGM*4), (Sup-) and (Und-) respectively entail*

> **(STriv)** *If* $A, B \in K$, *then* $A \rhd B \in K$
>
> **(UTriv)** *If* $A \in K$ *and* $B \notin K$, *then* $A \bowtie B \in K$

Proof of Observation 1. Regarding (Sup-) and (STriv): Assume that $A \in K$. By (Cl) and (Con) it follows that $\neg A \notin K$. Now further assumethat $B \in K$ and hence that $B \in \text{Cn}(K \cup \{A\})$. It follows by (AGM*4) that $B \in K * A$. By the right-to-left half of (Sup-), it then follows that $A \rhd B \in K$.

Regarding (Und-) and (UTriv): Assume that $A \in K$. It follows by (Cl) that $K = \text{Cn}(K \cup \{A\})$. Now further assume that $B \notin K$. It follows that $B \notin \text{Cn}(K \cup \{A\})$. By the contrapositive of (AGM*3), we then have $B \notin K * A$, and hence, by the right-to-left half of (Und-), $A \bowtie B \in K$ □

Observation 2. *(a) In the presence of (SVAC $\dot{-}$), (Sup) entails*

> **(RS)** *If* $B \notin K$, *then* $A \rhd B \in K$ *iff* $B \in K * A$

*(b) in the presence of (Cl), (AGM*3) and (AGM $\dot{-}$4), (Sup) is inconsistent with (STriv) on pains of entailing*

> ***(Triv1)*** *For all* $K \in \mathbb{K}$ *and non-tautologous* $A, B \in \mathcal{L}$, *if* $A, B \in K$, *then* $A \notin K \dot{-} B$

Proof of Observation 2. Regarding (a): Assume that $B \notin K$. By (SVAC $\dot{-}$), $B \in K * A$ iff $B \in (K \dot{-} B) * A$, and by (Sup), iff $A \rhd B \in K$.

Regarding (b): Assume (STriv). By (Sup), this is equivalent to the assumption that, for all $K \in \mathbb{K}$ and $A, B \in \mathcal{L}$, if $A, B \in K$, then $B \in (K \dot{-} B) * A$. Now assume that for some non-tautologous $A, B \in \mathcal{L}$, we have $A, B \in K$. It follows that $B \in (K \dot{-} B) * A$, and therefore, by (AGM*3), that $B \in \text{Cn}((K \dot{-} B) \cup \{A\})$. Finally, assume for reductio that $A \in K \dot{-} B$. It follows that $\text{Cn}((K \dot{-} B) \cup \{A\})$ is equal to $\text{Cn}(K \dot{-} B)$, and hence, by (Cl), to $K \dot{-} B$. So $B \in K \dot{-} B$. But since B was assumed to be non-tautologous, this contradicts (AGM $\dot{-}$ 4). □

Observation 3. *Given (Cl), (Con), (AGM*2), (AGM*3), (AGM*6), (AGM*7), (AGM $\dot{-}$3), (AGM $\dot{-}$5) and (DP $\dot{-}$4+), (Sup) entails:*

> **(SAbs)** *If* $A \rhd B \in K$, *then* $A \rhd (A \wedge B) \in K$
>
> **(SCA)** $A \rhd C \in K$ *and* $B \rhd C \in K$, *then* $(A \vee B) \rhd C \in K$
>
> **(SCC)** *If* $A \rhd B \in K$ *and* $A \rhd C \in K$, *then* $A \rhd (B \wedge C) \in K$
>
> **(SID)** *If* $A \notin Cn(\varnothing)$, *then* $A \rhd A \in K$
>
> **(SLLE)** *If* $(A \leftrightarrow C) \in Cn(\varnothing)$, *then* $A \rhd B \in K$ *iff* $C \rhd B \in K$.

(SHook) *If* $A \rhd B \in K$ *then* $A \rightarrow B \in K$

(SMP) *If* $A, A \rhd B \in K$*, then* $B \in K$

(SNec) *If* $\neg B \in Cn(\varnothing)$ *then* $A \rhd B \notin K$

(SRLE) *If* $(B \leftrightarrow C) \in Cn(\varnothing)$*, then* $A \rhd B \in K$ *iff* $A \rhd C \in K$

(SSCC) *If* $A \rhd B \in K$ *and* $\neg B \vee \neg C \in Cn(\varnothing)$*, then* $A \rhd C \notin K$

(Ent) *If* $A \notin Cn(\varnothing)$ *and* $B \in Cn(A)$*, then* $A \rhd B \in K$

Proof of Observation 3. Available on request.

Observation 4. *(a) In the presence of (SVAC*), (Und) entails*

(RU) *If* $B \in K$*, then* $A \bowtie B \in K$ *iff* $B \notin K * A$

(b), in the presence of (SVAC), (Und) is inconsistent with (UTriv) on pains of entailing*

(Triv2) *For all* $K \in \mathbb{K}$ *and non-contradictory* $A, B \in \mathcal{L}$*, if* $A \in K$ *and* $B \notin K$*, then* $A \notin K * B$

Proof of Observation 4. Regarding (a): Assume that $B \in K$. By (SVAC*), $B \notin K * A$ iff $B \notin (K * B) * A$, and by (Und), iff $A \bowtie B \in K$.

Regarding (b): Assume (UTriv). By (Und), this is equivalent to the assumption that, for all $K \in \mathbb{K}$ and $A, B \in \mathcal{L}$, if $A \in K$ and $B \notin K$ then $B \notin (K * B) * A$. Now assume that for some non-tautologous $A, B \in \mathcal{L}$, we have $A \in K$ and $B \notin K$. It follows that $B \notin (K * B) * A$. Now assume, for reductio, that $A \in K * B$. By (SVAC*), it then follows that $(K * B) * A = K * B$. It follows from the fact that B is non-contradictory, in conjunction with (AGM*2), that $B \in K * B$ and hence, by the previous equality, that $B \notin (K * B) * A$. Contradiction. □

Observation 5. *Given (AGM*2), (AGM*3), (AGM * V), (CM), (I*1), (AGM* $\dot{-}$*2), (AGM* $\dot{-}$*3), (AGM* $\dot{-}$*4), (AGM* $\dot{-}$*6), (DP* $\dot{-}$*2), (DP* $\dot{-}$*4), (Cl), (Sup) entails:*

(RRW) *If* $C \in Cn(B)$ *and* $A \rhd B \in K$*, then* $A \rhd C \notin K$ *iff* $A \rhd B \notin K \dot{-} C$.

Proof of Observation 5. See Chandler (2012), proof of Observation 5.

Observation 6. *Given (Cl), (Sup-) entails:*

(RW) *If* $C \in Cn(B)$ *and* $A \rhd B \in K$*, then* $A \rhd C \in K$

Proof of Observation 6. See Chandler (2012), proof of Observation 4.

Observation 7. *(Cl), (Con), (AGM*7), (AGM*8), (DP*1), (DP*2), (DP*3) and (DP*4) collectively entail:*

(1) *If* $C \in (K * C) * A$*, then* $B \in (K * C) * A$ *iff* $B \in K * C \wedge A$

(2) *If* $C, \neg C \notin (K * C) * A$*, then* $B \in (K * C) * A$ *iff* $B \in K * A \cap K * C \wedge A$

(3) *If* $\neg C \in (K * C) * A$*, then* $B \in (K * C) * A$ *iff* $B \in K * A$

Proof of Observation 7. Regarding (1): Assume $C \in (K * C) * A$. By (Cl) and (Con), we therefore have $\neg C \notin (K*C)*A$, and hence, by (AGM*7) and (AGM*8), $B \in (K * C) * C \wedge A$ iff $B \in \mathrm{Cn}((K * C) * A \cup \{C\})$. However, $\mathrm{Cn}((K * C) * A \cup \{C\})$ is itself equal, by (Cl) and the fact that $C \in (K * C) * A$, to $(K * C) * A$. So $B \in (K * C) * C \wedge A$ iff $B \in (K * C) * A$. But it follows from (DP*1) that $B \in (K * C) * C \wedge A$ iff $B \in K * C \wedge A$. Hence $B \in (K * C) * A$ iff $B \in K * C \wedge A$, as required.

Regarding (2): Assume (i) $C \notin (K * C) * A$ and (ii) $\neg C \notin (K * C) * A$. By (AGM*8), it follows from (i) and (ii) that if $B \in (K * C) * A$ then $B \in (K * C) * \neg C \wedge A \cap (K * C) * C \wedge A$. The converse implication follows from (AGM*7).[4] Therefore $B \in (K * C) * A$ iff $B \in (K * C) * \neg C \wedge A \cap (K * C) * C \wedge A$. By (DP*2), $B \in (K * C) * \neg C \wedge A$ iff $B \in K * \neg C \wedge A$. By (DP*1), $B \in (K * C) * C \wedge A$ iff $B \in K * C \wedge A$. So $B \in (K * C) * A$ iff $B \in K * C \wedge A \cap K * \neg C \wedge A$.

We now consider two further sub-cases: (a) $\neg C \in K * A$ and (b) $\neg C \notin K * A$.

(a) By (AGM*7) and (AGM*8), $B \in K * \neg C \wedge A$ iff $B \in \mathrm{Cn}((K * A) \cup \{\neg C\})$. From (a) and (Cl), it follows that $B \in \mathrm{Cn}((K * A) \cup \{\neg C\})$ iff $B \in K * A$ and hence that $B \in K * \neg C \wedge A$ iff $B \in K * A$. It then follows, from the fact that $B \in (K * C) * A$ iff $B \in K * C \wedge A \cap K * \neg C \wedge A$, that $B \in (K * C) * A$ iff $B \in K * C \wedge A \cap K * A$, as desired.

(b) From the fact that $C \notin (K * C) * A$, it follows by (DP*3) that we also have $C \notin K * A$. From this, (i), (AGM*7) and (AGM*8), we then recover $B \in K * A$ iff $B \in K * \neg C \wedge A \cap K * C \wedge A$. Since we have previously established that $B \in (K * C) * A$ iff $K * C \wedge A \cap K * \neg C \wedge A$, it follows that $B \in (K * C) * A$ iff $B \in K * A$. Finally, since, by (AGM*8), if $B \in K * A$ then $B \in K * A \wedge B$, we recover $B \in (K * C) * A$ iff $B \in K * C \wedge A \cap K * A$, as desired.

Regarding (3): Assume $\neg C \in (K * C) * A$. By a similar chain of reasoning to the one involved in (1) above, by (Cl), (Con), (AGM*7) and (AGM*8), it follows that $B \in (K * C) * \neg C \wedge A$ iff $B \in (K * C) * A$. Furthermore, by (DP*2), we have $B \in (K*C)*\neg C \wedge A$ iff $B \in K*\neg C \wedge A$. Putting these two biconditionals together yields $B \in (K * C) * A$ iff $B \in K * \neg C \wedge A$. From $\neg C \in (K * C) * A$, it follows by (DP*4) that $\neg C \in K * A$. As we have already shown (see case (1)(a) above), it follows from this, given (Cl), (AGM*7) and (AGM*8), that $B \in K * \neg C \wedge A$ iff $B \in K * A$. Since we have previously established that $B \in (K * C) * A$ iff $B \in K * \neg C \wedge A$, it then follows that $B \in (K * C) * A$ iff $B \in K * A$, as desired. □

Observation 8. *In the presence of (Cl), (Con), (SVAC), (AGM*2), (AGM*3), (AGM*V), (AGM*7), (AGM*8), (DP*1), (DP*2) and (DP*3), (Und) entails:*

[4] To see why, note that (AGM*7) entails: If $C \in K * A$ and $C \in K * B$, then $C \in K * A \vee B$ (see proposition (3.14) of [Gär88]).

(RRS) *If* $B \in \mathrm{Cn}(C)$ *and* $A \bowtie B \in K$*, then* $A \bowtie C \in K$ *iff* $A \bowtie B \in K * C$

Proof of Observation 8. Given (Und), (RRS) is equivalent to:

(RRS') If $B \in \mathrm{Cn}(C)$ and $B \notin (K * B) * A$, then $C \notin (K * C) * A$ iff $B \notin ((K * C) * B) * A$.

We assume the antecedent of (RSS'), i.e. $B \in \mathrm{Cn}(C)$ and $B \notin (K * B) * A$, throughout. Now either (i) $\neg C \in \mathrm{Cn}(\varnothing)$ or (ii) $\neg C \notin \mathrm{Cn}(\varnothing)$.

Assume (i). It immediately follows by (Con) that $C \notin (K * C) * A$. Since, given (AGM*V), (i) entails $K * C = K$, it follows from $B \notin (K * B) * A$ that $B \notin ((K * C) * B) * A$. So we have both $C \notin (K * C) * A$ and $B \notin ((K * C) * B) * A$, from which the consequent of (RRS') obviously follows.

So now assume (ii). We first derive the right-to-left direction of the consequent of (RSS'). Assume $B \notin ((K*C)*B)*A$. From (ii), by (AGM*2) we have $C \in K*C$ and hence, by (Cl) and the fact that $B \in \mathrm{Cn}(C)$, we recover $B \in K * C$. Given this, by (SVAC), it then follows that $(K * C) * B = K * C$ and hence, from the fact that $B \notin ((K * C) * B) * A$, that $B \notin (K * C) * A$. But since $B \in \mathrm{Cn}(C)$, we recover, by (Cl), the desired conclusion that $C \notin (K * C) * A$.

Regarding the left-to-right direction: Assume $C \notin (K * C) * A$. From the fact that $B \notin (K*B)*A$, by (DP3), it follows that $B \notin K*A$. Assume $B \in ((K*C)*B)*A$ for reductio. As noted above, given our background assumptions, it follows from (ii) and the fact that $B \in \mathrm{Cn}(C)$, that $(K*C)*B = K*C$. So $B \in ((K*C)*A$. In the light of parts (2) and (3) of Observation 7, it then follows from this, in conjunction with the fact that $C \notin (K * C) * A$, that $B \in K * A$. Contradiction. □

Observation 9. *Given (Cl), (Und-) entails:*

(RS) *If* $B \in Cn(C)$ *and* $A \bowtie B \in K$*, then* $A \bowtie C \in K$

Proof of Observation 9. Assume that (i) $B \in \mathrm{Cn}(C)$ and (ii) $A \bowtie B \in K$. By the left-to-right half of (Und-) and (ii), it follows that $B \notin K * A$. From this, by (i) and (Cl), we have $C \notin K * A$. By the right-to-left half of (Und-), it then follows that $A \bowtie C \in K$. □

Observation 10. *In the presence of (Cl), (AGM*2), (AGM*3), (AGM*4), (AGM*V), (AGM*7), (AGM*8), (SVAC*), (SVAC $\dot{-}$), (DP*1), (DP*2), (DP*3), (DP*4) and (DP $\dot{-}$4), (Und) and (Sup) jointly entail:*

If (1) $B \rhd C \in K$*, (2)* $A \bowtie (B \rhd C) \in K$*, (3)* $C \notin K$ *and (4)* $A \rhd C \notin K$*, then* $A \bowtie C \in K * B$*.*

Proof of Observation 10. Assume (1)–(4). It follows from (2) and (Und) that $B \rhd C \notin (K * B \rhd C) * A$, and hence, given (Sup), that $C \notin (((K * B \rhd C) * A) \dot{-} C) * B$. Furthermore, given (1) and (SVAC*), the latter is equivalent to:

(5) $C \notin ((K * A) \dot{-} C) * B$

Now note that, from (4) and (Sup) it follows that $C \notin (K \dot{-} C) * A$. And from the latter, in the presence of (3) and (SVAC $\dot{-}$), we can derive:

(6) $C \notin K * A$.

Given (6) and (SVAC $\dot{-}$), (5) is equivalent to:

(7) $C \notin (K * A) * B$.

Now assume for reductio that $\neg A \in (K * A) * B$. It follows from part (3) of Observation 7 that if $C \in K * B$ then $C \in (K * A) * B$. Given (1), (Sup) and (DP $\dot{-}$4), we have

(8) $C \in K * B$

Hence $C \in (K * A) * B$. But this contradicts (7). So $\neg A \notin (K * A) * B$. Now either (i) $A \in (K * A) * B$ or (ii) $A \notin (K * A) * B$. Assume (i). It then follows from part (1) of Observation 7 that if $C \in K * A \wedge B$, then $C \in (K * A) * B$. By (7), it therefore follows that $C \notin K * A \wedge B$. Assume (ii). It then follows from part (2) of Observation 7 that if $C \in K * B$ and $C \in K * A \wedge B$, then $C \in (K * A) * B$. By (7) and (8), it therefore follows, once again, that $C \notin K * A \wedge B$. So we can conclude:

(9) $C \notin K * A \wedge B$.

Now either (i) $A \notin (K * B) * A$ or (ii) $A \in (K * B) * A$. Assume (i). It follows from parts (2) and (3) of Observation 7 that if $C \in (K * B) * A$, then $C \in K * A$. By (6), we then recover $C \notin (K * B) * A$. Assume (ii). It follows from part (1) of Observation 7 that if $C \in (K * B) * A$, then $C \in K * A \wedge B$. By (9), we then again recover $C \in (K * B) * A$. So, either way, $C \notin (K * B) * A$. But, given (8), it follows from this, by (SVAC*), that $C \notin ((K * B) * C) * A$. Finally, given (Und), this last claim is equivalent to $A \bowtie C \in K * B$. □

Observation 11. *Given (SVAC*) and (DPf3), (Sup) and (Und) entail that if* $B \bowtie (A \triangleright B) \in K$*, then* $A \triangleright B \notin K$

Proof of Observation 11. Assume, for reductio, that (1) $A \triangleright B \in K$ and (2) $B \bowtie (A \triangleright B) \in K$. Given (Sup) and (Und), (2) amounts to (3) $A \triangleright B \notin (K * A \triangleright B) * B$. But (3) is equivalent, given (1) and (SVAC*), to $A \triangleright B \notin K * B$, which is itself equivalent, given (Sup), to (4) $B \notin ((K * B) \dot{-} B) * A$. However, given (DPf3), (4) entails $B \notin (K \dot{-} B) * A$. Contradiction. □